J. Phys. A: Math. Gen. 28 (1995) 1535–1547. Printed in the UK

Critical behaviour of self-avoiding walks that cross a square

Neal Madras†

Department of Mathematics and Statistics, York University, 4700 Keele Street, Downsview, Ontario, Canada M3J 1P3

Received 26 July 1994

Abstract. Consider the set of all self-avoiding walks in the square lattice which start at (0, 0), end at (L, L), and are entirely contained in the square $[0, L] \times [0, L]$. Associate a fugacity x with each step of the walk. Whittington and Guttmann (1990) showed that the dominant walks have $O(L)$ steps when x is small and $O(L^2)$ steps when x is large, and they conjectured that there is a single transition point at $x = \mu^{-1}$, where μ is the inverse of the connective constant for (unconstrained) self-avoiding walks. We present a rigorous proof of this conjecture (and its analogue in higher dimensions). We also discuss what can be said rigorously about two scaling exponents associated with this phase transition, and compare this with analogous results that have been obtained exactly (and rigorously) on the discrete Sierpinski gasket by Hattori, Hattori and Kusuoka (1990).

1. Introduction

The self-avoiding walk has long been a standard model of a long linear polymer molecule in a good solvent (Madras and Slade 1993). The usual setting is a single walk on an infinite lattice, which models a polymer in a dilute solution. If we consider instead a (large) finite region of a lattice, then the solution changes from dilute to dense as we increase the length of the walk (or the fugacity for the number of steps). In fact, Whittington and Guttmann (1990) proved the existence of a dilute-to-dense phase transition for the model described in the next paragraph. The aim of the present paper is to prove some rigorous results about this transition.

To fix ideas, let us begin with self-avoiding walks on the square lattice $\mathbb{Z}^2$. For large L, consider the set of all self-avoiding walks which start at the origin $(0, 0)$, end at (L, L), and are entirely contained in the square $[0, L] \times [0, L]$. Associate a fugacity x with each step of the walk. Whittington and Guttmann (1990) showed that when x is small the dominant walks have $O(L)$ steps, while when x is large the dominant walks have $O(L^2)$ steps. They showed that the transition occurred for x somewhere between μ^{-1} and μ_{H}^{-1}, where μ is the connective constant for (unconstrained) self-avoiding walks and μ_{H} is the connective constant for Hamiltonian walks in a square. They conjectured, on numerical grounds, that there was a single transition point at μ^{-1}. This conjecture was supported by a renormalization analysis (Prentis 1991) and by a correspondence with N-vector models (Burkhardt and Guim 1991).

In this paper we give a rigorous proof of the conjecture of Whittington and Guttmann (1990). We also discuss what can be said rigorously about two scaling exponents associated

† E-mail address: madras@nexus.yorku.ca

0305-4470/95/061535+13$19.50

susceptibility $\chi(z)$ for $\mathbb{Z}^2$ (see equation (15)). So our theorem 2.1(ii) shows that the critical fugacity x_b^* of $C_b^{\mathrm{ZMS}}(x)$ converges to μ^{-1} as $b \to \infty$ (see table 2 and footnote [23] in Živić *et al* (1993)). (We remark that the squares in Živić *et al* (1993) are rotated by 45°, but that does not affect our methods.)

Two questions regarding the dense phase ($x > \mu^{-1}$) were left conspicuously unanswered in the present paper, namely, can we say anything rigorous about the belief that the limiting free energy f_2 behaves as $(x - \mu^{-1})^{d\nu}$ as x decreases to μ^{-1}? And can we prove the existence of the limiting free energy for dense walks with free endpoints? These appear to be hard questions which deal directly with detailed properties of dense walks.

There is one more intriguing question about the dense phase: can we prove the existence of a limiting *probability distribution* for any of the ensembles described in this paper, for any $x > \mu^{-1}$? This would give a natural measure on a class of *infinite, dense* polymers. We believe that this would be easier for animals or trees than for walks. For one very special case, this has actually been accomplished by Pemantle (1991): he considered uniformly distributed spanning trees of $[-L, L]^d$, corresponding to $x = +\infty$ in our tree models. Pemantle showed that these distributions have a weak limit as $L \to \infty$, but the limiting objects are trees only for $d \leqslant 4$; for $d \geqslant 5$, they are *disconnected*, so the limiting distribution is actually on spanning forests of $\mathbb{Z}^d$. We do not know whether this surprising fact has any analogue when $\mu^{-1} < x < +\infty$, or whether something similar happens for walks or animals.

Acknowledgment

This research was supported in part by an operating grant from the Natural Sciences and Engineering Research Council of Canada.

References

Burkhardt T W and Guim I 1991 *J. Phys. A: Math. Gen.* **24** L1221–8
Chayes J T and Chayes L 1986 *Critical Phenomena, Random Systems, and Gauge Theories (Les Houches, Session XLIII, 1984)* ed K Osterwalder and R Stora (Amsterdam: Elsevier) pp 1001–142
de Gennes P G 1979 *Scaling Concepts in Polymer Physics* (Ithaca, NY: Cornell University Press)
Golowich S E and Imbrie J Z 1993 *Preprint*
Hammersley J M 1957 *Proc. Camb. Phil. Soc.* **53** 642–5
Hara T and Slade G 1992 *Commun. Math. Phys.* **147** 101–36
Hattori K and Hattori T 1991 *Prob. Theor. Rel. Fields* **88** 405–28
Hattori K, Hattori T and Kusuoka S 1990 *Prob. Theor. Rel. Fields* **84** 1–26
Hattori T and Kusuoka S 1992 *Prob. Theor. Rel. Fields* **93** 273–84
Klarner D A 1967 *Can. J. Math.* **19** 851–63
Klein D J 1981 *J. Chem. Phys.* **75** 5186–5189
Klein D J and Seitz W A 1984 *J. Physique Lett.* **45** L241–7
Madras N and Slade G 1993 *The Self-Avoiding Walk* (Boston: Birkhäuser)
Pemantle R 1991 *Ann. Probab.* **19** 1559–74
Prentis J J 1991 *J. Phys. A: Math. Gen.* **24** 5097–103
Rammal R, Toulouse G and Vannimenus J 1984 *J. Physique* **45** 389–94
Saleur H 1987 *Phys. Rev.* B **35** 3657–60
Spitzer F 1976 *Principles of Random Walks* (New York: Springer)
Whittington S G and Guttmann A J 1990 *J. Phys. A: Math. Gen.* **23** 5601–9
Živić I, Milošević S and Stanley H E 1993 *Phys. Rev.* E **47** 2430–9